Recap

HW 1 posted
Due : 10/16

Bases

- Every vector space has a basis
- All bases of a finitely generated space V have same size : $\dim(V)$

Linear Transformations

V, W over same $\mathbb{F}$. $\varphi : V \to W$ such that

- $\varphi(v_1) + \varphi(v_2) = \varphi(v_1 + v_2)$
- $\varphi(c \cdot v) = c \cdot \varphi(v)$

e.g. $A \in \mathbb{F}^{m \times n}$, $\frac{d}{dx} : \mathbb{R}[x] \longrightarrow \mathbb{R}[x]$, $\varphi_{left} : Fib \longrightarrow Fib$

Specifying a linear transformation

- V, W vector spaces over some $\mathbb{F}$. Basis B for V.

Let $\alpha : B \longrightarrow W$ be any function. Then $\exists$ a unique LT

$\varphi : V \longrightarrow W$ s.t. $\varphi(v) = \alpha(v) \;\; \forall v \in B$

<u>Proof:</u>

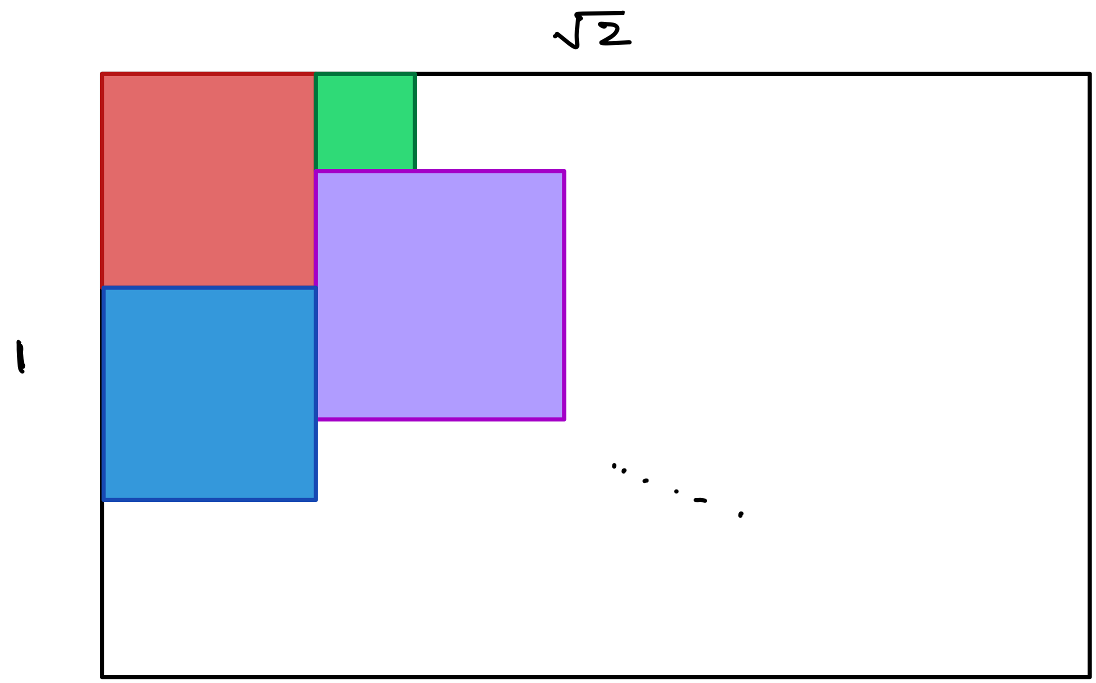
$v \in V$ Let $B = \{w_1, \ldots \ldots w_k\}$

$$v = c_1 \cdot w_1 + \cdots - - - + c_k \cdot w_k$$

$$\varphi(v) = c_1 \cdot \underbrace{\varphi(w_1)}_{=\alpha(w_1)} + \cdots + c_k \cdot \underbrace{\varphi(w_k)}_{=\alpha(w_k)}$$

$$\varphi(v) = c_1 \cdot \alpha(w_1) + \cdots + c_k \cdot \alpha(w_k)$$

Interlude: a linear algebra puzzle [Matousek]



Can the rectangle be filled by a finite number of non-overlapping squares?

Hint: Vector space $\mathbb{R}$ over $\mathbb{Q}$. Consider linear $\varphi: \mathbb{R} \to \mathbb{R}$

s.t. $\varphi(1) = 1$ and $\varphi(\sqrt{2}) = -1$.

Kernel and Image

$\varphi : V \longrightarrow W$

- $\ker(\varphi) = \{ v \in V \mid \varphi(v) = 0 \} \subseteq V$
- $\operatorname{im}(\varphi) = \{ \varphi(v) \mid v \in V \} \subseteq W$

Ex: Show that $\ker(\varphi)$, $\operatorname{im}(\varphi)$ are subspaces.

e.g. $A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 2 \end{bmatrix}$ $\qquad \varphi_A : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$

$$\operatorname{im}(\varphi_A) = \operatorname{Span}\left\{ \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \end{pmatrix} \right\} = \mathbb{R}^2$$

$$\ker(\varphi_A) = \operatorname{Span}\left\{ \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} \right\}$$

Rank-nullity theorem

- Let $\varphi : V \longrightarrow W$ be a linear transformation, for finite dimensional V. Then.

$$\underbrace{\dim(\operatorname{im}(\varphi))}_{\text{rank}(\varphi)} + \underbrace{\dim(\ker(\varphi))}_{\text{nullity}(\varphi)} = \dim(V) \Rightarrow n \text{ (say)}$$

Proof: Say $\{v_1 \ldots v_k\}$ are a basis for kernel

Complete to basis $\{v_1, \ldots v_k, v_{k+1}, \ldots, v_n\}$ for V

$$\underbrace{\varphi(v_{k+1}) \cdots \varphi(v_n)}$$

must form basis for $\operatorname{im}(\varphi)$

A useful kernel

$$A = \begin{bmatrix} 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 \end{bmatrix} \in \mathbb{F}_2^{3\times 7} \qquad \varphi_A : ? \to ?$$

- $\text{rank}(\varphi_A) = \dim(\text{im}(\varphi_A)) = 3$
- $\text{nullity}(\varphi_A) = \dim(\ker(\varphi_A)) = 4$
- $\ker(\varphi_A)$ is an error-correcting code.
  Can fix any one-bit mistake.

Eigenvalues and eigenvectors

$\underset{\text{operators}}{\underbrace{\varphi : V \longrightarrow V}}$ $\qquad \lambda \in \mathbb{F}$ is called an eigenvalue if

$\exists\, v \in V \setminus \{0_V\}$ s.t. $\varphi(v) = \lambda \cdot v$

(eigenvector)

Spectrum : $\operatorname{Spec}(\varphi) = \{\lambda \in \mathbb{F} \mid \lambda$ is an eigenvalue of $\varphi\}$

e.g. $\begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} : \mathbb{R}^2 \longrightarrow \mathbb{R}^2 \qquad \operatorname{Spec}(\varphi) = \{0, 2\}$

$\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} : \mathbb{R}^2 \longrightarrow \mathbb{R}^2 \qquad \operatorname{Spec}(\varphi) = \emptyset$

$\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} : \mathbb{C}^2 \longrightarrow \mathbb{C}^2 \qquad$ Ex:

Examples without matrices

$$\varphi : C^\infty([0,1], \mathbb{R}) \longrightarrow \underset{\sim\sim\sim\sim\sim}{C^\infty([0,1]}, \mathbb{R})$$

↳ infinitely differentiable functions

$$\varphi(f) = \frac{df}{dx} \qquad \operatorname{Spec}(\varphi) = \mathbb{R}$$

$$\frac{d}{dx} e^{\lambda x} = \lambda \cdot e^{\lambda x}$$

Ex: $\varphi_{\text{left}} : \mathbb{R}^{\mathbb{N}} \longrightarrow \mathbb{R}^{\mathbb{N}}$ $\qquad \varphi_{\text{left}}(f)(n) = f(n+1)$

$$\operatorname{Spec}(\varphi) = \ ?$$

Eigenspaces

- Let $U_\lambda = \{v \in V \mid \varphi(v) = \lambda \cdot v\}$. Then $\forall \lambda \in \mathbb{F}$,
  U_λ is a subspace of V.

Proof:

$$\varphi(v_1 + v_2) = \varphi(v_1) + \varphi(v_2)$$

$$= \lambda \cdot v_1 + \lambda \cdot v_2$$

$$= \lambda \cdot (v_1 + v_2)$$

$\dim(U_\lambda)$ is called geometric multiplicity of λ.

Independence of eigenvectors

- Let $\lambda_1, \dots, \lambda_k$ be distinct eigenvalues of φ, and let $v_1, \dots, v_k$ be associated eigenvectors. Then the set $\{v_1, \dots, v_k\}$ is LI.

Proof: If not let $v_{i_1}, \dots, v_{i_r}$ be smallest LD set

$$c_1 \cdot v_{i_1} + \dots + c_r \cdot v_{i_r} = 0 \qquad (1)$$

$$\downarrow \varphi$$

$$c_1 \cdot \lambda_{i_1} v_{i_1} + \dots + c_r \cdot \lambda_{i_r} \cdot v_{i_r} = 0 \qquad (2)$$

$\lambda_{i_r} \cdot (1) - (2)$ gives

$$\{ c_1 \cdot (\lambda_{i_r} - \lambda_{i_1}) \cdot v_1 + \dots + c_{r-1} \cdot (\lambda_{i_{r-1}} - \lambda_{i_r}) \cdot v_{i_{r-1}} = 0$$

even smaller dependency!